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Original Contribution

APPLICATION OF POLYNOMIAL TRANSFORMATION IN GEOREFERENCE OF GRAPHIC CADASTRAL PLANS

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Abstract: *Polynomial transformation is a classic mathematical tool in geodesy, used to convert coordinates between different reference systems. It is mainly used in cases where the exact geometric or physical model of the relationship between the systems is unknown, as well as in the presence of significant local deformations in the source data. In practice, this method is used in: transformation between coordinate systems (including local), approximation of heights when creating geoid models, localization in GNSS measurements, as well as in specific tasks in photogrammetry and laser scanning. One of its most widespread applications remains the georeferencing of raster images.*

This paper investigates the specifics of applying polynomial transformation for georeferencing purposes. The main emphasis is placed on the dependence of the results on the configuration and the number of control points used. An analysis is made and recommendations are formulated for their optimal distribution on the image. The mathematical requirement for uniform grid coverage in a minimum of n+1 columns and rows (where n is the degree of the polynomial) is justified as a condition for avoiding singularity. Additionally, the negative effects of the collinearity of the control points on the computational stability are considered, as well as the approaches for identifying the Runge phenomenon (appearance of high-amplitude oscillations) in the extrapolation zones along the boundaries of the transformed range.

Key words: *Affine transformation, Polynomial transformation, Georeferencing, Cadastral plans, Collinearity, Runge phenomenon.*

1. Introduction

The cadastral plan is a reduced, orthogonal, detailed image of a part of the earth's surface, which is created through geodetic measurements. The main elements of the cadastral plan are the type, location, purpose and boundaries of real estate. In the past, the cadastral plan had a heterogeneous content, the data in it had different meanings and was created for different territories. It is produced on a non-deformable basis (sheets) as a map with symbols and signs. A register (record sheet) is created for objects (properties) with the plan, in which data about the properties are recorded, including data on substantive law [2].

The cadastral plan has a general geographical content: relief, infrastructure, waters and watercourses, vegetation cover and others. In the period 1949 - 2001, the cadastral plan is essentially a large-scale topographic plan. It depicts the situation and relief, water and technical infrastructure at the stage of development of the plan. In terms of property, cadastral plans contain the boundaries of land properties (if they existed on site) - identified by plan numbers and outlines of buildings with their signatures (type, construction, number of floors).

Similar to topographic maps, when depicting large territories, cadastral plans are arranged on sheets next to each other, forming a grid of rectangles and squares - a grid. Each map sheet is named - a nomenclature. The grid of the plans of the settlements is unique for each. For example, the grid of the plan sheets of Sofia covers over 1200 sheets at a scale of 1:1000 with dimensions of 50x80 cm. The number of the cadastral plan and its neighbors are written on the frame of the sheet.

Often, a number of cadastral plans have been created for the same territory in different periods of time. Each reproduction of the territory in a cadastral plan is called an issue, and the map sheets of the corresponding edition may have been created in different years. The cadastral plan is a basic graphic document for the objects in the urbanized territory. According to the Law on the Unified Cadastre of the People's Republic of Bulgaria (LUCRB), cadastral plans are prepared with different thematic content and cadastral units (plan of underground conduits and facilities, general plans of industrial enterprises, forest management plans, etc.). These thematic plans, after the entry into force of the Law on the Cadastre and Property Register (LPRC), are called specialized maps.

During the years since the liberation of the Bulgarian state, the creation of cadastral plans has been regulated by various normative acts, including those created for the purposes of urban planning, such as the Law on the Planned Development of Settlements (LPDS) and the Law on Territorial and Settlement Planning (LTSP) [10]. During the period of the LPDS and LTSP (1949–2001), the cadastral plan was developed, adopted and periodically supplemented by design organizations. Even after the adoption of the LUCBR (1980), the main emphasis of the cadastral plans at that time was on regulation and urban planning, with significantly less attention paid to the registration and description of property.

The mathematical basis of the cadastral plans is the skeleton, which guarantees accuracy and the possibility of joining individual map sheets into a common geodetic system. Over the years, various geodetic and reference systems have been used to create cadastral plans, the main ones being: Bulgarian Geodetic System 1930 (BGS30), Bulgarian Geodetic System 1950 (BGS50) and Coordinate System 1970 (CS). Regardless of the different practical implementation of the individual coordinate and reference systems, the basic principle for drawing the map content remains the same – by using the so-called “coordinate grid”, which is usually a grid of squares with a side of 10 cm, drawn on the map sheet (canvas). To draw this grid, precise instruments such as coordinate graphs (mechanical devices with magnifying glasses and screws) were used to draw the grid crosses with an accuracy of up to 0.1 mm. In addition to drawing the map content, the coordinate grid also serves to control the deformations of the paper/cardboard [9].

Despite the global digitalization of modern spatial data, the use of analog graphic materials remains a technological and legal necessity, determined by the status of the previous cadastral plans. The reason is that various phenomena, structures and facilities, rights, restrictions, etc., are directly related to precisely this type of graphic images, which necessitates their processing. In many cases, there are also requirements in the regulatory framework that explicitly refer to the type of graphic materials in question. This article examines some features of the application of polynomial transformation, implemented using the coordinate grid of cadastral plans

2. Essence of polynomial transformation

Polynomial transformation is a mathematical method used primarily in geodesy, photogrammetry, and GIS to correct geometric distortions and georeferencing raster images. It establishes a mathematical relationship between two coordinate systems through polynomial equations [1].

The general formula for a two-dimensional polynomial transformation of degree n describes the relationship between two coordinate systems (***x, y***) and (***X, Y***) through the following system of equations:

$$\begin{aligned} X &= \sum_{i=0}^{n} \sum_{j=0}^{i} a_k \cdot x^{i-j} \cdot y^{j} \\ Y &= \sum_{i=0}^{n} \sum_{j=0}^{i} b_k \cdot x^{i-j} \cdot y^{j} \end{aligned}, \quad (1)$$

where:

- ***(x, y)*** are the source coordinates (e.g. pixels in an image);
- ***(X, Y)*** are the transformed coordinates (the destination coordinates);
- $\boldsymbol{a_k}$, $\boldsymbol{b_k}$ are the polynomial coefficients that are calculated using control points;
- ***n*** is the degree of the polynomial.

The presence of degrees in this type of transformation requires the determination of the minimum number of necessary coefficients for each of the corresponding degrees of the polynomial:

$$N = \frac{(n+1)(n+2)}{2}, \quad (2)$$

where ***N*** is the number of coefficients for each of the two coordinates, and ***n*** is the degree of the polynomial. In this case, the final number of coefficients required will be ***2 x N***. Since each selected point defines two coefficients (these are its plan coordinates – ***X*** and ***Y***), then ***N*** coincides with the minimum number of necessary control points used in the transformation. An example of the required number of points and parameters is shown in Table 1.

Table 1.

Degree (n)	Transformation type	Coefficients (for one axis)	Общо за X и Y	Min. number of points
1	Linear (Affine)	3	6	3
2	Quadratic	6	12	6
3	Cubic	10	20	10
4	Quartic	15	30	15
5	Quintic	21	42	21

The mathematical apparatus for solving polynomial transformations is based on the compilation of systems of correction equations, through which the unknown transformation parameters (coefficients of the polynomial) are determined. Since in practice the number of identical (support) points used almost always exceeds the minimum required for a unique solution, the calculation is carried out strictly by applying the method of least squares (MLS). This approach requires the compilation and solution of a system of normal equations in matrix

form, which allows both the extraction of the most probable values of the coefficients and the reliable assessment of the accuracy of the results. In general, matrix equations are expressed by the following dependence:

$$X = (A^T PA)^{-1} A^T PL \,, \tag{3}$$

where:

- $\boldsymbol{X}$ is the column vector of the sought transformation coefficients. They define the specific mathematical model of the selected polynomial, describing the translation, rotation, scaling and possible nonlinear deformations in the transition from the pixel coordinate system of the raster to the target geodetic coordinate system;
- $\boldsymbol{A}$ is the structural matrix (matrix of the plan), the elements of which are formed by the local (pixel) coordinates of the selected identical (control) points, read directly on the digital image;
- $\boldsymbol{A^T}$ is the transposed matrix of $\boldsymbol{A}$;
- $\boldsymbol{P}$ is the weight matrix, which in standard georeferencing procedures is most often taken as a single matrix ($\boldsymbol{P} = \boldsymbol{I}$), based on the assumption that the localization of all control points is equal. If a differentiated approach is needed (for example, due to different reliability in the visual identification of individual points), the corresponding weights are set along the main diagonal of the matrix;
- $\boldsymbol{L}$ is the column vector of the free terms, composed of the known spatial (geodesic) coordinates of the same control points in the target reference system.

After determining the most probable values of the transformation parameters, the vector of residual errors (corrections) is calculated using the relationship $\boldsymbol{V} = \boldsymbol{AX} - \boldsymbol{L}$. This vector is the basis for calculating the root mean square error (RMS), which serves as the main quantitative criterion for assessing the geometric accuracy of the georeferencing performed and the reliability of the resulting final raster [4].

3. The problem of collinearity of control points in polynomial transformation

As mentioned, the determination of the unknown polynomial coefficients $\boldsymbol{a_k}$ $\boldsymbol{and}$ $\boldsymbol{b_k}$ is performed by the method of least squares (LMS) by composing a system of linear equations [4].

$$L = AX \,, \tag{4}$$

where $\boldsymbol{A}$ is the matrix of plan coefficients. To obtain a unique solution, it is necessary to invert the matrix of normal equations $N=A^T PA$ (with equal weights: $N=A^T A$. A mathematical requirement for the existence of a stable and unique solution is that the matrix $\boldsymbol{A}$ has full rank, and the determinant of the normal matrix is nonzero ($\det(N)\neq 0$) [8].

The phenomenon of mathematical singularity (unsolvability of the system) occurs when the columns of the matrix A become linearly dependent [11]. The geometric condition for this is formulated as follows: **the system falls into singularity if all selected control points lie on an algebraic curve of degree lower than or equal to the degree of the sought polynomial transformation**. In the case of an affine transformation ($n = 1$), the system is singular if all points lie on a straight line (algebraic curve of first degree: $A_x + B_y + C = 0$). In the case of a polynomial of second degree ($n = 2$), the matrix becomes invertible if the points lie on a conic section or its degenerate forms, such as two parallel or two intersecting straight lines [12].

When georeferencing cadastral plans, the most reliable and unambiguously identifiable contours of the image are the crosses (nodes) of the decimeter coordinate grid. In its essence,

it represents a strict orthogonal structure with a constant interval (usually 100 m in the scale of the plan). However, this geometric regularity hides a high risk of unintentionally introducing linear dependencies [8].

If the control points are selected only along the grid lines (rows and columns), rigid algebraic dependencies of the form $(Y - Y_1)(Y - Y_2) = 0$ are formed. To avoid this, it is necessary to define a strict geometric range constraint: for a polynomial of degree n, the control points must necessarily be spatially distributed over at least $n + 1$ different rows and $n + 1$ different columns of the coordinate grid [12]. The recommended spatial distribution parameters are systematized in the following table 2:

Table 2.

Degree of polynomial (*n)*	Minimum number of parameters	Minimum required number of points	Recommended number of points	Minimum geometric range in the network	Minimum configuration area
1 (affine)	6	3	4-6	2 x 2 lines	100 x 100 m (1 square)
2 (quadratic)	12	6	9-12	3 x 3 lines	200 x 200 m (4 squares)
3 (cubic)	20	10	16-20	4 x 4 lines	300 x 300 m (9 squares)

4. Experimental studies

An important aspect of the theoretical framework is the distinction between pure algebraic solvability and numerical stability of computer algorithms [6]. It is appropriate to consider the limiting case of a polynomial of degree 2 with 6 available control points, located in a configuration of 2 rows and 3 columns on a 100-meter grid. This is the most common case of images representing a fragmentary sample of a given graphic cadastral plan in A4 format. In this condition, there is a strict dependence of degree 2 between the points ($Y^2 - 300Y + 20000 = 0$) and the system is in a pure singularity ($det(N) = 0$).

Hypothetically, introducing a perturbation (divergence) of even just one of these 6 points in a direction orthogonal to the line of the row ($\Delta Y \neq 0$) removes the perfect linear dependence. Theoretically, for a displacement of the order of 1 mm ($\Delta Y = 0.001$ m), the matrix $\boldsymbol{A}$ acquires full rank, the determinant becomes nonzero, and the system obtains a mathematical solution. However, in computational practice, this minimal discrepancy generates **an ill-conditioned matrix**, in which the number of conditions tends to infinity [8]. The consequences of an ill-conditioned system in geodetic practice are:

- unnaturally large polynomial coefficients that seek to mutually compensate for unstable geometry [6, 11];
- extreme sensitivity to errors – a minimal error in pixel selection (even at the subpixel level) is reflected in deformations of the final coordinates of the order of tens of meters;
- Runge phenomenon – the appearance of sharp twists and uncontrollable wave-like oscillations of the model along the periphery and outside the contour of the control points [4, 5].

For the purposes of this experiment, a part of a vector drawing of a cadastral plan was printed on an engineering canvas (transparent polyester film) scanned in raster format. The main idea is to perform the experiment in a controlled environment. The scope of the fragmentary sample is 300 / 300 m, or 4 x 4 coordinate crosses at a scale of 1:1000 and a distance between coordinate crosses of 100 m. The scanning of the drawing was performed with a resolution of 600 dpi (1 pixel = 0.042 m), and the recording format is TIFF. Fig. 1

shows the drawing, and Fig. 2 schematically shows the reported deformations between the coordinate crosses, which are expected to be due to the technical characteristics of the printing device, the imperfections of the scanning device, or a combination of both factors. However, the values of the deformations have no practical impact for the purposes of this experiment.

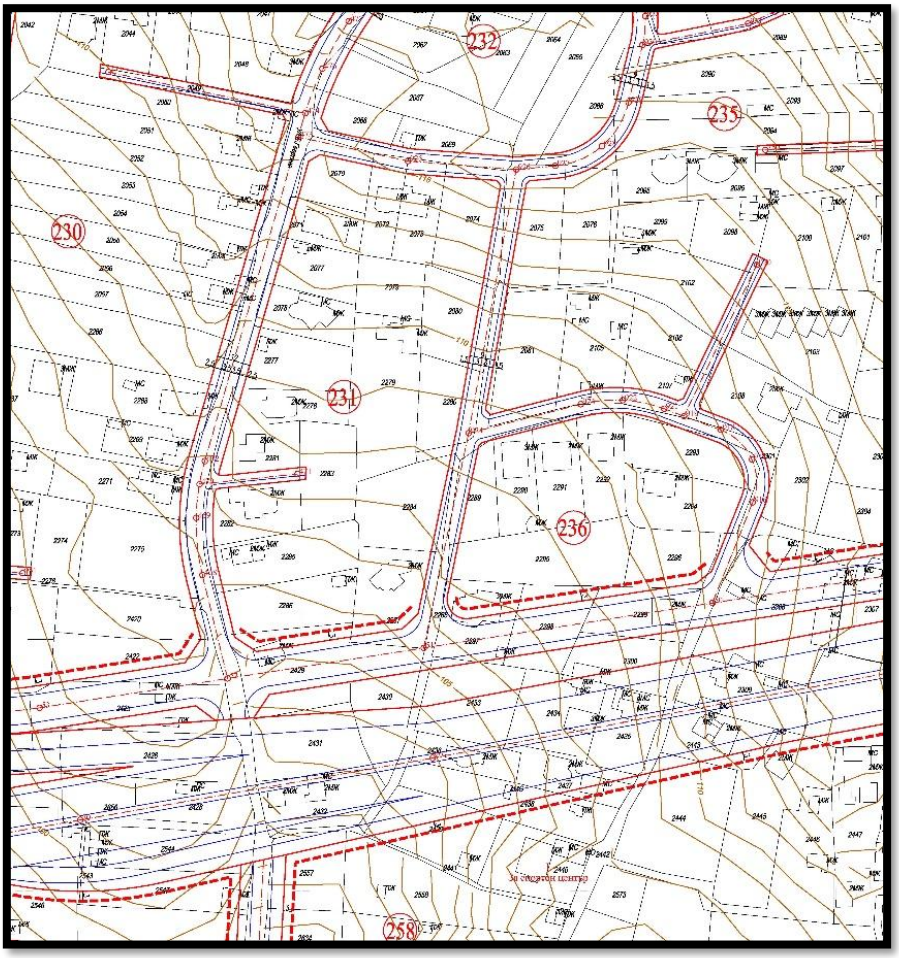


Fig. 1. Vector drawing used for the present experiment

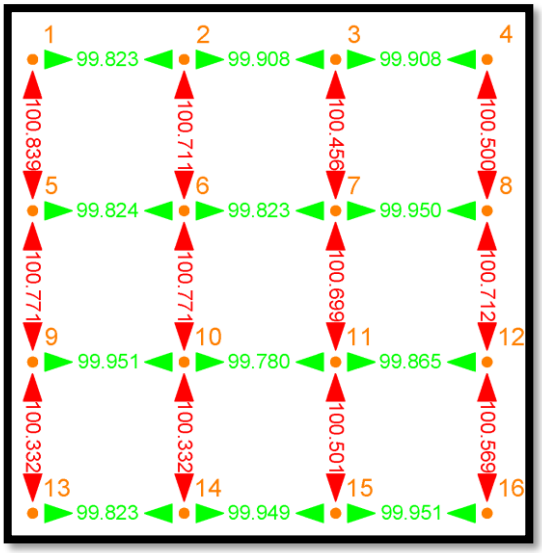


Fig. 2. Schematic representation of the deformations between the coordinate crosses

The green vectors indicate local deformations of the "compression" type of the coordinate grid, while the red vectors depict deformations of the "expansion" type.

Initially, an attempt is made to perform a second-degree polynomial transformation, specifying the minimum number of required points (6) in the form of three columns and two rows (corresponding coordinate crosses in Fig. 2 – 6, 7, 8, 10, 11 and 12). Although the requirement for the minimum number of points required for a second-degree polynomial transformation is theoretically met, as well as the minimum discrepancy regarding the collinearity of the control points, the software does not allow the transformation to proceed (Fig. 3).

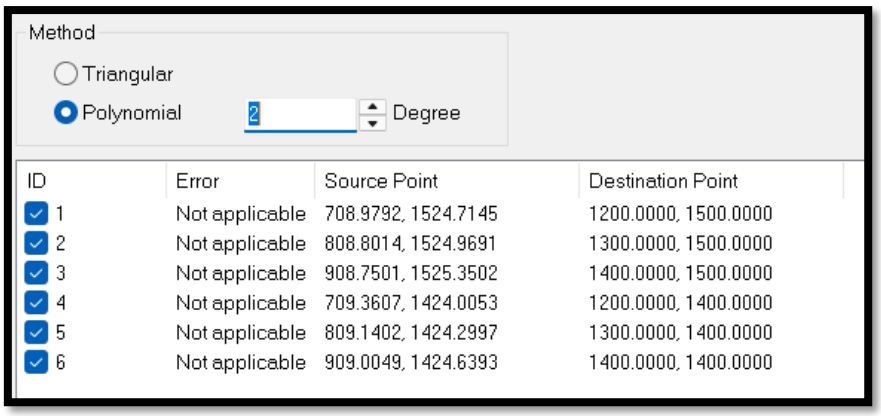


Fig. 3. Results of an attempt to transform a sample of a cadastral and regulatory plan in the AutoCAD Raster Design programming environment (two rows, three columns)

Subsequently, another control point is added along the first row, corresponding to coordinate cross No. 5 from Fig. 2. The results of the transformation are shown in Fig. 4. Despite the seemingly low values of mean square errors in the used control points, the image is deformed to the point of unusability.

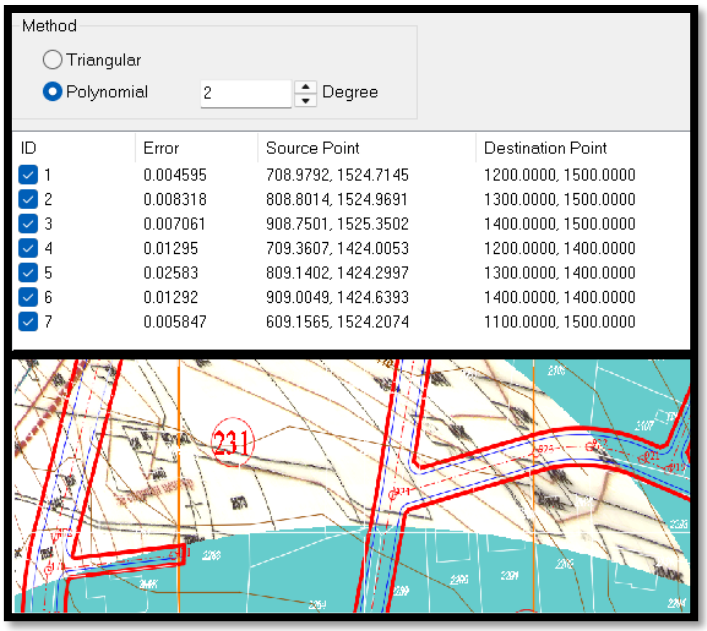


Fig. 4. Results of conducting a second-degree polynomial transformation within the scope of the cadastral and regulatory plan sample in the AutoCAD Raster Design programming environment (two rows, four columns)

Then, a new experiment was performed, in which the condition that the control points should be spatially distributed over a minimum of n+1 different rows and n+1 different columns of the coordinate grid (points №2, 5, 6, 7 and 10 in Fig. 2) was satisfied. Theoretically, the same can be done with five control points, but they do not cover the necessary parameters for conducting a second-degree polynomial transformation. For the purposes of the experiment, a sixth control point was added, which is not in accordance with the columns and rows generated by the already selected five control points (point №16 in Fig. 2). Since six is the minimum number of required points, practically no LMS adjustment is performed and the mean square errors in the control points are equal to zero (Fig. 5).

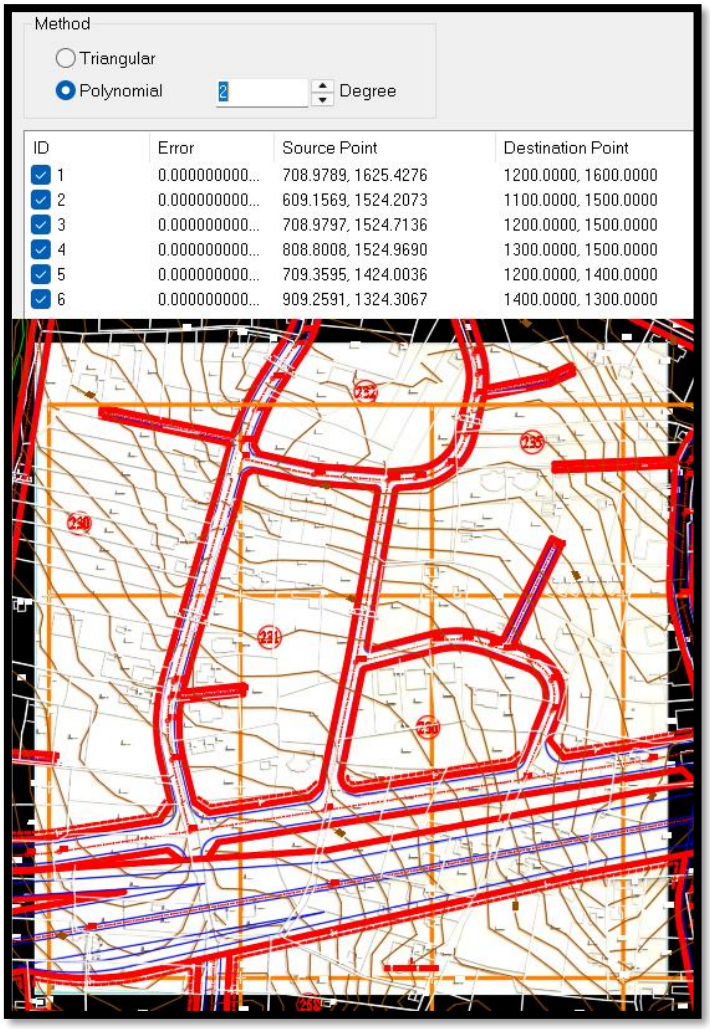


Fig. 5. Results of applying a second-degree polynomial transformation to the sample scope of the cadastral and regulatory plan in the AutoCAD Raster Design programming environment (three rows, three columns)

Despite the larger deviations obtained in the areas of unused control points, the image in Fig. 5 has a seemingly usable appearance, provided that the condition for a minimum number of columns and rows is observed. These results necessitate an experiment to confirm the possibility of performing a polynomial transformation according to the example in Fig. 3 (two rows and three columns) and, accordingly, the value of the necessary displacement of the sixth point, so that the image has a seemingly usable appearance. The latter sets the condition that there should be no larger internal deformations between the control points in the centers of the two squares between the control points.

A series of second-degree polynomial transformations were performed, with the location of the sixth point being shifted incrementally by 1 meter with each subsequent transformation. Summary results are shown in Tables 3 and 4 and Figs. 6 and 7. Table 3 – moving point #7 outside (south) the area of interest.

Table 3. Moving point #7 outside (south) of the area of interest

Moving point №7 from Fig. 2	Deviation to the left - points № 1, 2, 5 and 6 of Fig. 2	Deviation to the right - points № 2, 3, 6 and 7 of Fig. 2
1 m	3.88 m	4.24 m
2 m	1.99 m	2.31 m
3 m	1.30 m	1.56 m
5 m	0.77 m	1.06 m
10 m	0.36 m	0.72 m
15 m	0.30 m	0.48 m
20 m	0.25 m	0.44 m
25 m	0.20 m	0.42 m
30 m	0.18 m	0.38 m
40 m	0.13 m	0.36 m
50 m	0.11 m	0.35 m

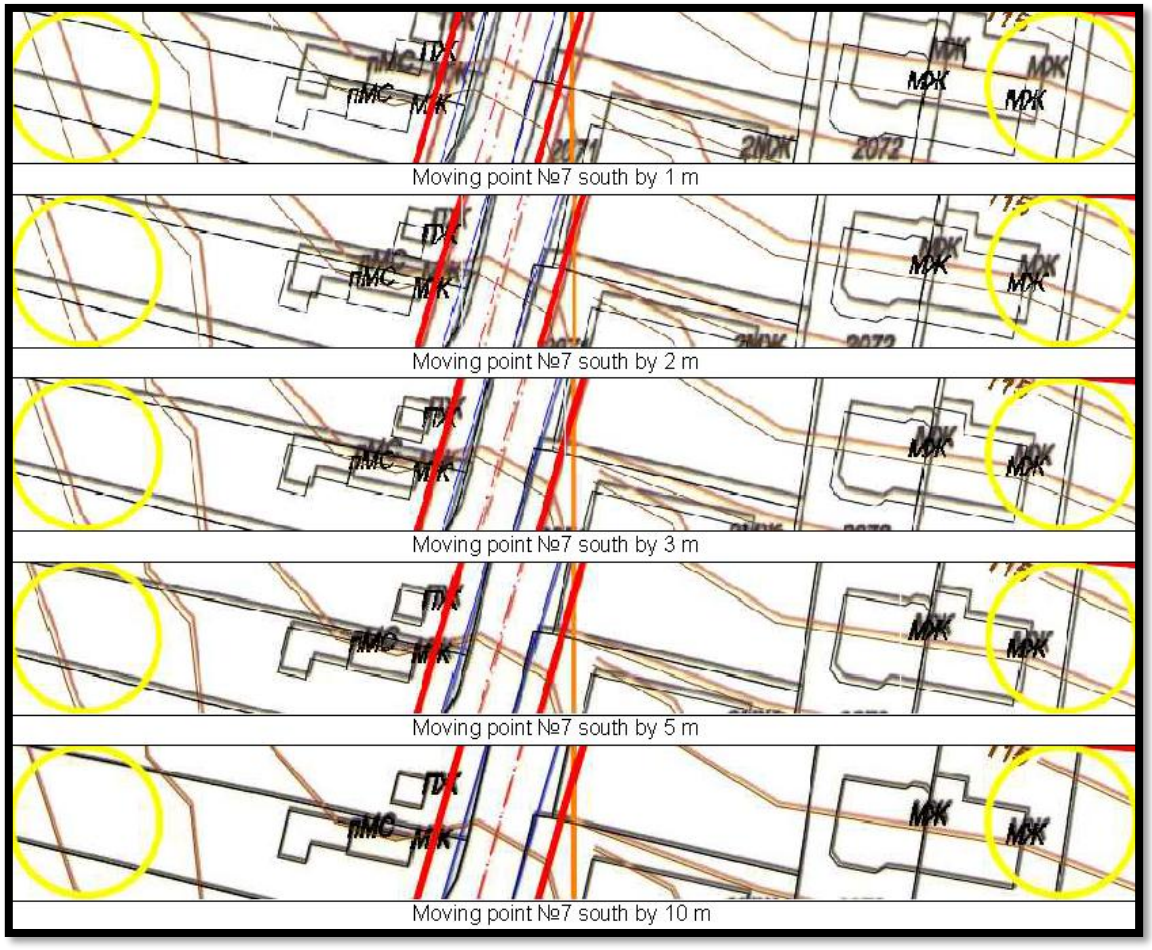


Fig. 6. Graphical representation of the displacement outside the area of interest (south) of point No. 7 from the experiment performed

Table 4. Moving point №7 towards (north) the area of interest

Moving point №7 from Fig. 2	Deviation to the left - points № 1, 2, 5 and 6 of Fig. 2	Deviation to the right - points № 2, 3, 6 and 7 of Fig. 2
1 m	3.74 m	3.65 m
2 m	1.93 m	1.62 m
3 m	1.12 m	0.92 m
5 m	0.75 m	0.48 m
10 m	0.36 m	0.15 m
15 m	0.25 m	0.12 m
20 m	0.18 m	0.11 m
25 m	0.14 m	0.10 m
30 m	0.10 m	0.12 m
40 m	0.10 m	0.22 m
50 m	0.18 m	0.28 m

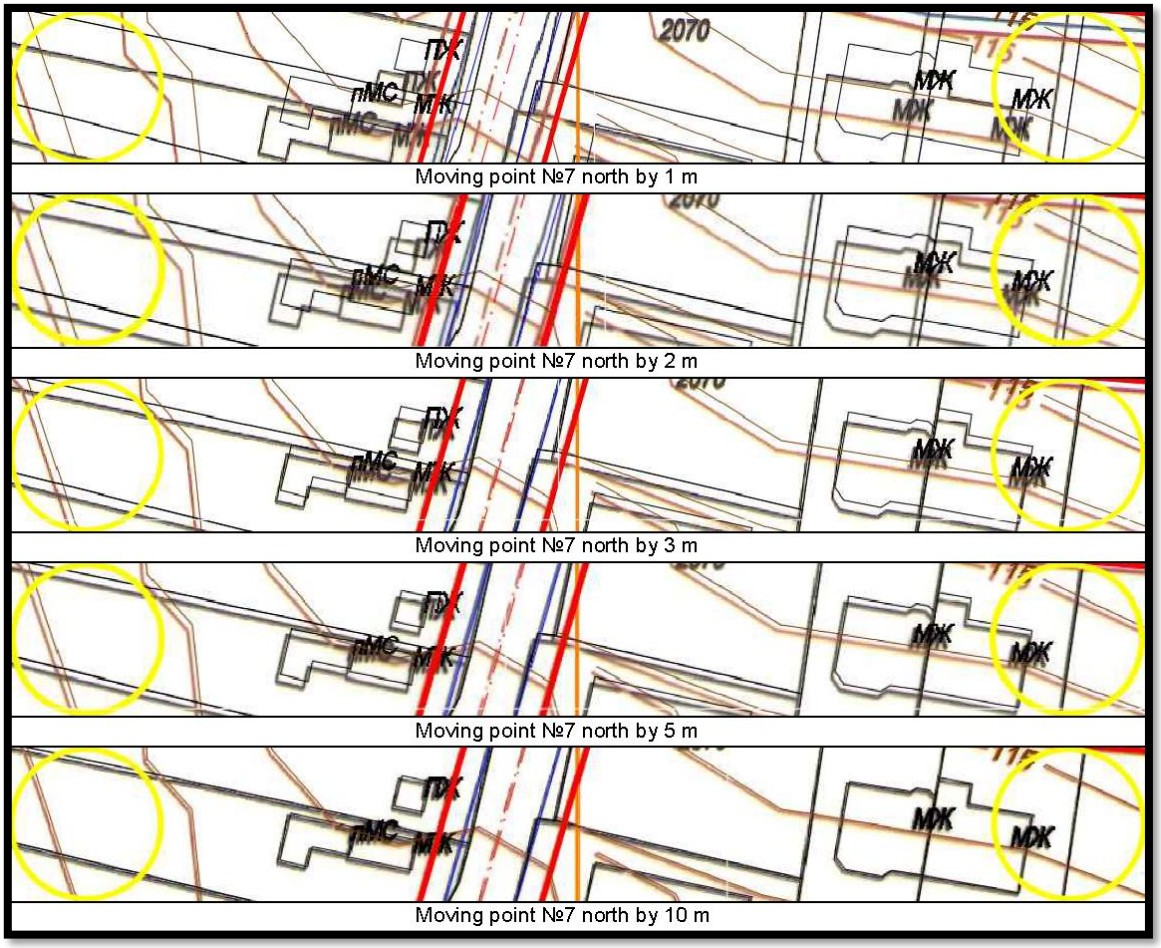


Fig. 7. Graphical representation of the displacement in the direction of the area of interest (north) of point No. 7 from the experiment performed

The comparative analysis of the results presented in Table 3 and Table 4 reveals a clear asymmetry in the gradient of the residual errors depending on the direction of translation of control point No. 7. Larger deformations are observed when the point is moved outside the area of interest, in the extrapolation area.

The specific numerical data unambiguously illustrate this trend. With a translation of 5 m to the south (extrapolation), the deviation in the right part of the studied range (affecting points No. 2, 3, 6 and 7) is 1.06 m. For comparison, with an equivalent translation of 5 m to the north, directed towards the interior of the network, the same deviation is more than half as small – only 0.48 m. This slow rate of error decay when moving outwards is also preserved at larger translations; for example, at 10 m to the south the deviation is 0.72 m, while at 10 m to the north it drops to 0.15 m.

The reason for this behavior is rooted in the extrapolation nature of the polynomial function. Since the mathematical model of a second-degree transformation includes quadratic terms (x2 and y2), the influence of the error or offset increases proportionally to the square of the distance from the origin of the coordinate system or centroid. When a control point is shifted beyond the range of the others, it falls into an extrapolation zone and begins to act as a point with high statistical leverage.

In this peripheral zone, a deficiency of support nodes occurs, which allows the polynomial function to generate extreme, uncontrollable, and high-frequency oscillations. Due to the nonlinear nature of the quadratic terms in the coordinate transformation, the model is forced to generate large curvature to satisfy the equation. This leads to extreme deformations of the image outside the convex hull of the support points, an effect analogous to the Runge phenomenon in numerical analysis. Conversely, when translating north, the point moves to an interpolation zone closer to the centroid, where the mathematical influence of the quadratic terms is geometrically limited and stabilized by the neighboring control points, leading to a faster decrease in deformations.

5. Conclusion

The problem of local deformations in polynomial transformations of degree 2 and higher on a nearly collinear network arises from the mathematical instability of poorly conditioned matrices [4]. When a control point is shifted outside the range of the others (extrapolation zone), it begins to act as a point with high statistical leverage. Due to the nonlinear nature of the quadratic terms in the coordinate transformation, the model is forced to generate a large curvature to satisfy the equation, which leads to extreme deformations of the image outside the convex hull of the control points - an effect analogous to the Runge phenomenon in numerical analysis [12].

Therefore, to guarantee real numerical stability in the transformation, the intentional or technological deviation of the points from the main lines must be macroscopic, and based on the conducted experiment, **a recommended shift of more than 15% when the point is in the opposite direction to the others and more than 10% when it is in their direction can be proposed**. Only then is the perturbed point capable of effectively defining an independent third order in space. These percentage values **should be considered as heuristic thresholds related** to the specifics of the type of map images under consideration.

In real conditions, these recommendations are difficult to apply, due to the specificity of this type of graphic images. In order to add a control point that satisfies the above recommendation, it is necessary for it to be individualized in the graphic image and for it to have known plan coordinates (for example, a point from a geodetic base) or a characteristic object from the graphic image to be coordinated on the terrain, thus serving as a control point. The latter is associated with other risks, such as the state of the absolute accuracy of the

graphic image, which directly depends on the geodetic networks built at the time of its creation and the errors contained in them, as well as possible tectonic movements [3]. An effective method for assessing the relative accuracy of the cadastral plan and adding additional control points can also be achieved by the methods discussed in [7].

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