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Original Contribution

EVALUATION OF THE QUALITIES OF LAMBERTIAN AND GAUSSIAN PROJECTIONS

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Abstract: *One of the important characteristics of a projection is the scale factor m of the image, which is a measure of the deformation of the projection and generally depends on the location and direction.*

Key words: *Scale, Image, Deformation.*

1. Introduction

From the point of view of mathematics, in all projections we have, generally speaking, an "image". In addition to scale, an important qualitative indicator is the simplicity of the law of representation [1, 2].

An important question that can be raised is whether the projection should be conformal? In the case of conformal projection, the angles are kept undeformed during the depiction. Nowadays, considerations are put forward according to which more distances than angles are measured, which requires the preservation of their true values. In the preparation of cadastral maps and in some spatial activities, it is important to preserve the areas of the individual districts, i.e. an equivalent (equilateral) projection is used, which is not conformal. The following quality of the conformal projection should be emphasized, that the scale depends only on the place, but not on the direction. Therefore, the preference for conformity of the image is preserved [1, 2, 4, 5, 9, 12].

2. Lambert projection

The normal conformal conic projection is associated with the name of the German mathematician and philosopher Johann Heinrich Lambert (1728–1777), whose work „Anmerkungen und Zusatze zur Entwerfung der Land – und Himmelscharten“, published in 1772, preceded the works of Lagrange and Gauss in the field of conic projection. Therefore, the projection in question is also called the conic projection of Lambert, or Lambert's projection. At a later stage, the projection found application in countries such as Belgium and France and in some states of the United States. In Bulgaria, the Lambert projection has found a well-deserved place both in the educational literature and in the creation of the Bulgarian Geodetic System 2000 (BGS 2000) [1, 2, 4, 5, 6, 9, 11, 16].

In the conic Lambert projection, the selected central meridian does not coincide in meaning with the central meridian of the Gaussian (and also UTM) projection, but is usually determined by the mean ellipsoidal longitude of the depicted territory [1, 2, 12].

Therefore, for the difference in lengths l we will have

$$l = \lambda - \lambda_0 \text{, the radius of the parallel e} \quad r = \frac{a.\cos\varphi}{\left(1 - e^2.\sin^2\varphi\right)^{1/2}} \tag{1}$$

where φ and λ are the ellipsoidal geographic coordinates, λ_0 is the longitude of the selected central meridian [4, 5, 9, 15].

The isometric width will be equal to

$$q = \frac{1}{2}\ln\left[\frac{(1+\sin\varphi)}{(1-\sin\varphi)}\right] + \frac{1}{2}e.\ln\left[\frac{(1-e.\sin\varphi)}{(1+e.\sin\varphi)}\right] \tag{2}$$

he definitional equations of projection are

$$\begin{aligned} x &= X_0 - \rho.\cos(\alpha.l) \\ y &= Y_0 + \rho.\sin(\alpha.l) \end{aligned}, \tag{3}$$

where (x, y) are the plane-shaped coordinates, and the radius-vector ρ will be determined by the formula

$$\rho = C.\exp(-\alpha.q). \tag{4}$$

The definition parameters are α <1 and C >0.

The constant X_0 can be selected in such a way that the abscissance is positive, for example provided that $X_0 > \rho$.

The scale *m* is determined by the formula

$$m = \frac{\alpha.C.\exp(-\alpha.q)}{\rho}. \tag{5}$$

At a given meridian $\lambda = const$ the equations in variable ρ will form a line that passes through the same center and which forms a straight circular cone. A tangent or split cone can be used, which is unfolded in the plane (shaped), relative to the earth's rotational ellipsoid, whose axis coincides with the axis of the cone [1, 2, 6, 12].

The parallel along which the scale is equal to 1 is called "standard" and sometimes "main". From a geometrical point of view, this will mean that the points of this parallel are points from the cone. In this case, the cone is tangent to the ellipsoid along this standard parallel. When we have two standard parallels, the cone will cross the ellipsoid exactly along them. Standard parallels can be selected or determined on the basis of some criterion of optimality [1, 2, 6, 7, 8, 11, 12].

The wide application of conical conformal projection confirms its good qualities (provided that it is applied to a small width variation range).

When choosing the best projection for a given territory, it must be said that image deformations are inevitable. The projection in question is conformal, therefore, in the image, the angles are preserved, the scale depends only on the place, but not on the direction. To a lesser or greater extent, depending on the difference in scale by unit, lengths and faces are deformed [1, 2, 6, 7, 8, 11, 12].

Usually, the optimization of the projection is associated with the appropriate selection of parameters α and *C*, which will lead to small deformations of the mapped territory. However, optimization (in a broader sense) can be carried out according to pre-introduced requirements, which are not necessarily subject to the condition for the best distribution of deformations throughout the territory. For example, stripes can be selected (based on preliminary considerations) by which the image will be as plausible as possible or the projection can be described with the simplest possible equations. In both cases, the issue of the distribution of deformations will remain in the background, but nowadays the analytically defined projection allows, in view of the great capabilities of computing, to reach the elements of the reference surface (the earth ellipsoid) relatively easily [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14].

To determine the locations of the intersecting parallels and calculate the value of the scale *m* in a conformal conical projection with two standard parallels, we will use the formulas:

$$-\frac{d\rho}{R.d\varphi}=\frac{\alpha.\rho}{R.\cos\varphi} \quad \Rightarrow \quad \frac{d\rho}{\rho}=-\alpha.\frac{d\varphi}{\cos\varphi} \tag{6}$$

The requirement is that the image must be conformal, i.e. $m=n$ and $\varepsilon=0$.

After integration, it is located

$$\ln\rho=-\alpha.\ln tg\left(45^0+\frac{\varphi}{2}\right)+\ln C \quad \Rightarrow \quad \rho=\frac{C}{tg^{\alpha}\left(45^0+\frac{\varphi}{2}\right)}, \tag{7}$$

where *C* is an integration constant.

For the equator, at $\varphi=0$, it is found that $C=\rho_{eq.}$, which means that the constant C represents the radius of the image of the equator. Substituting for the radius ρ it turns out

$$\rho=\frac{\rho_{eq.}}{tg^{\alpha}\left(45^0+\frac{\varphi}{2}\right)}=\frac{\rho_{eq.}}{V^{\alpha}} \tag{8}$$

$$n=\frac{\alpha.\rho_{eq.}}{V^{\alpha}.R.\cos\varphi}=\frac{\alpha.C}{r.V^{\alpha}}, \qquad 1=\frac{\sin\varphi_t.\rho_{eq.}}{V^{\alpha}.R.\cos\varphi_t} \quad \Rightarrow \quad \rho_{eq.}=R.\cot g\varphi_t.V^{\alpha}$$
$$m=n, \qquad p=n^2, \qquad \omega=0. \tag{9}$$

Rigorous method of determining coefficients α and C when cutting a cone.

$$n_1 = n_2 = 1, \qquad \frac{\alpha.C}{r_1.V_1^{\alpha}} = \frac{\alpha.C}{r_2.V_2^{\alpha}}, \qquad r_1.V_1^{\alpha} = r_2.V_2^{\alpha} \tag{10}$$

$$\lg r_1 + \alpha.\lg V_1 = \lg r_2 + \alpha.\lg V_2, \quad \alpha.\left(\lg V_1 - \lg V_2\right) = \lg r_2 - \lg r_1 \quad \Rightarrow \ \alpha = \frac{\lg r_2 - \lg r_1}{\lg V_1 - \lg V_2} \tag{11}$$

$$C = \frac{r_1.V_1^{\alpha}}{\alpha}, \qquad C = \frac{r_2.V_2^{\alpha}}{\alpha} \tag{12}$$

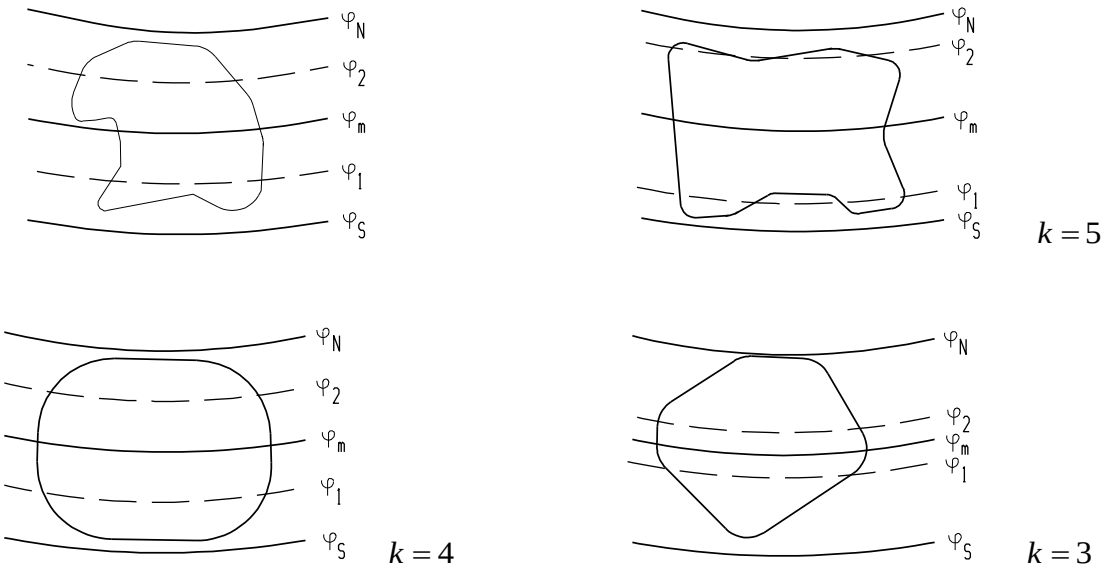


Fig. 1. Determination of the Locations of Standard Parallels in a Cutting Cone

$$\varphi_m = \frac{1}{2}\left(\varphi_S + \varphi_N\right), \qquad \varphi_1 = \frac{1}{2}\left(\varphi_S + \varphi_m\right), \qquad \varphi_2 = \frac{1}{2}\left(\varphi_m + \varphi_N\right) \tag{13}$$

$$\varphi_1 = \varphi_S + \frac{\varphi_S + \varphi_N}{k}, \qquad \varphi_2 = \varphi_N - \frac{\varphi_S + \varphi_N}{k} \tag{14}$$

In this case, the coefficient *k* depends on the shape of the territory depicted.

For the territory of Bulgaria:

We choose $\varphi_N = 44^0 15^0$ and $\varphi_S = 41^0 10`$ $\Rightarrow$ $\varphi_1 = 41^0 47`$ and $\varphi_2 = 43^0 38`$

We accept the places of the intersecting parallels to be $\varphi_1 = 41^0 30`$ и $\varphi_2 = 43^0 30`$.

$\varphi_{avg} = 42^0 30`$, $R = \sqrt{M.N} = 6367,247$ km, $M = 6341,985$ km, $N = 6392,609$ km,

$r_1 = 4768,786$ $V_1 = 2,21991768280269$

$r_2 = 4618,638$ $V_2 = 2,32756303939659$

$\alpha = 0,675624513386098$

$C_1 = \rho_{eq.} = 12097,51366$ km, $C_2 = 12097,51366$ km.

Based on the calculations made, we will give the following comparative tables (Table 1, 2, 3). In order to visually see the changes in scales, they are calculated at intervals of $0^0 30`$.

Table 1. Amendment of the scale

Паралел (φ)	41^0	$41^0 30`$	42^0	$42^0 30`$	43^0	$43^0 30`$	44^0	$44^0 30`$
Мащаб (m)	1,000189	**1**	0,999886	0,999848	0,999885	**1**	1,000192	1,000462

Scale values m for a conformal conical (Lambert) projection with two standard parallels (Table 2) and one standard parallel (Table 3), determined by formulas from [15].

Latitude of the south standard parallel – $\varphi_1 = 41^0 51`11,2153``$, latitude of the north standard parallel – $\varphi_2 = 43^0 28`35,8786``$, latitude of the middle parallel – $\varphi_0 = 42^0 40`00``$.

Table 2. Amendment of the scale

φ	$41^0 21`11,21``$	$41^0 51`11,2153``$	$42^0 21`11,21``$	$42^0 51`11,21``$	$43^0 28`35,8786``$	$43^0 58`35,87``$
m	1,000126	**1**	0,99988	0,99987	**1**	1,000128

Table 3. Amendment of the scale

Паралел (φ)	$41^0 10`$	$41^0 40`$	$42^0 10`$	$42^0 40`$	$43^0 10`$	$43^0 40`$	$44^0 10`$
Мащаб (m)	1,000339	1,000151	1,000038	**1**	1,000038	1,00015	1,00034

3. Gaussian and UTM projections

The Gaussian projection can also be joined by the famous UTM (Universal Transverse Mercator) projection. They are the same type of cylindrical projections, differing only in name. The beginning of the coordinate system is taken the intersection of the equator with one of the meridians, called the main one for the particular system. The scale along the entire prime meridian for a given system has a constant value, which is equal to one or a value close to one. As you move away from the prime meridian, the deformations in the image will increase [1, 2, 6, 11, 12, 15, 16].

The Earth is divided into three- and six-degree Gaussian coordinate systems (stripes, zones), which extend from pole to pole and have a scale along the prime meridian $m_0 = 1$. In three-degree stripes, the prime meridian is more distant than $1,5^0$ to the east and west. The stripes are numbered without interruption, with the number *n = 1* having the stripe with a prime meridian $L_0 = 3^0$, number *n = 2* – the stripe with the prime meridian $L = 6^0$ etc. At six-degree strips, the prime meridian is farther away than 3^0 To the east and west, the stripes are also numbered without interruption, with the number *n = 1* having the one with a prime meridian $L_0 = 3^0$, number *n = 2* – the stripe with the prime meridian $L = 9^0$ etc. The relationship between the coordinate system number and the central meridian $L = L_0$ is shown in Fig. 2. From the graph it can be seen that every second prime meridian of the three-degree stripes coincides with the main meridian of the six-degree strip [1, 2, 6, 11, 12, 15, 16].

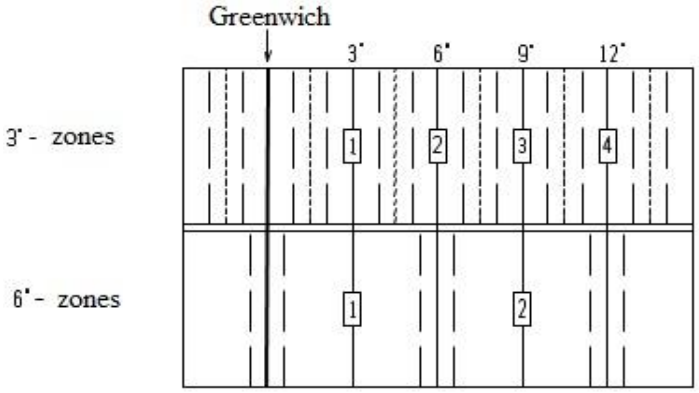


Fig. 2. Gaussian zones

Table 4. Number of zones

3^0 - ови ивици	$L_0 =$	0^0	3^0	6^0	9^0	12^0	15^0	18^0	21^0	.....
$n = L_0/3, L_0 = 3n$	$n =$	0	1	2	3	4	5	6	7	.....

Table 5. Number of zones

6^0 - ови ивици	$L_0 =$	3^0	9^0	15^0	21^0	27^0	33^0	.....
$n = \frac{1}{6}(L_0 + 3)$, $L_0 = 6n - 3$	$n =$	1	2	3	4	5	6	.....

The abscissa *X* is in the direction of the prime meridian, and the ordinate *Y* starts from the *X* axis – in the east it is positive, in the west it is negative. In order not to get the ordinates negative, Baumgart's proposal to add 500 000 m = m to them was accepted, and the last digit of the coordinate system number, i.e. n.10^6 m, came as the millionth digit.

$$X = x,$$
$$Y = Y_0 + y = \left(n.10^6 + 5.10^5\right) + y. \quad (15)$$

From the way the ordinate *Y* is written, it is evident the location of a point with respect to the principal meridian. If the digit following the millionth digit is less than 5, the point has a negative ordinate *y*, considered from the prime meridian, and is therefore located to the west of it, and if it is 5 or above 5, it has a positive ordinal y and lies to the east of it [1, 2, 6, 11, 12, 15, 16].

The territory of Bulgaria is covered by:

- Two 3^0 gaussian coordinate systems with prime meridians $L_0 = 24^0$ and $L_0 = 27^0$ and respectively numbers 8 and 9;
- Two 6^0 gaussian coordinate systems with fundamental meridians $L_0 = 21^0$ and $L_0 = 27^0$ and with numbers 4 and 5. The scale is equal to one along the prime meridian and grows to the east and west of it. It can be seen that a meridian of length $L_0 = 27^0$ is repeated.

UTM (Universal Transverse Mercator). This name of the Gaussian projection is imposed according to the terminology adopted by the US Military Geographic Service, and is also accepted in NATO countries as one of their standards. It is used to create military topographic maps on a small scale - from 1:50 000 upwards. It is applied only on 6^0 stripes with a scale along the main meridian m_0 = 0,9996. The numbering of the stripes is carried out according to the same rules as those described above, only that the beginning, instead of the 3^0 from the Greenwich Meridian (0^0+3^0) is put on 3^0 from meridian $L=180^0\left(180^0+3^0\right)$, i.e. on the reverse side of the same meridian ellipse. Calculated the numbers in which the territory of our country falls, you will get [1, 2, 6, 11, 12, 15, 16]:

$n=\left(180+L_0+3\right)/6$, where L_0 is the longitude of the prime meridian.

About Bulgaria : $L_0=21^0$, $n=\left(180+21+3\right)/6=34$, $L_0=27^0$, $n=\left(180+27+3\right)/6=35$.

The numbers of the stripes in this form now become inconvenient to be "embedded" with the values of the ordinate *Y*. Here only the added value of 500 000 will remain to the ordinate. In this situation, the unambiguous determination of the location of a point by its *X, Y* coordinates, and the indication of the respective zone in which the point falls. In the implementation of UTM, the coordinate locations have been reversed, with the ordinate *Y* placed in the first position, with the designation E (Easting), and in the second position – the abscissa *X* with the designation N (Norting). This arrangement is purely formal, but it facilitates contact with the different CAD systems [1, 2, 6, 11, 12, 15, 16].

With the adoption of scale along the prime meridian m_0 = 0,9996 will lead to another redistribution of the linear deformations, starting with a shortening along the main meridian $m<1$, $\delta=m-1<0$, go through a scale value $m = 1$ and linear deformation $\delta=0$ and then stretching at the border (along the boundary meridian) of the strip $m>1$, $\delta=m-1>0$, (Fig. 3).

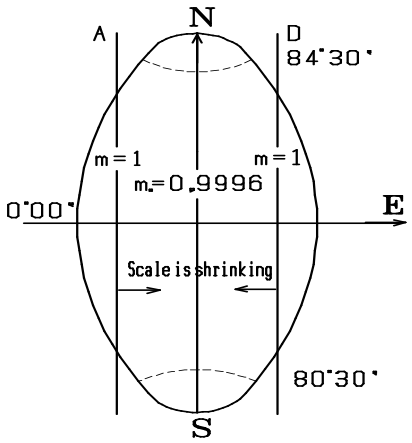


Fig. 3. Linear deformations

UTM coordinates occupy a special place among the other projection coordinates. Firstly, as projection coordinates – they are the same type for each of the stripes and in this sense it does not matter in the vicinity of which meridian they are defined. Secondly, in the way the beginning of the system and the numbers of the stripes are defined, only the geodetic longitude is sufficient λ_i, on which the GNSS receiver is located in order to calculate the

measurements at the site φ_i , λ_i coordinates in the UTM type. In this situation, without submitting any additional information through the UTM coordinates, any place on the globe can be determined [1, 2, 6, 12, 15, 16].

It should not be forgotten that for this purpose, in addition to the coordinates of the respective place (E, N), the number of the zone, i.e. (E, N, Zon), should also be present. For the wider range of geodetic, cartographic and navigation tasks that are solved on the territory of the country, UTM coordinates would be more appropriate. This applies to works related to geodetic working networks, photographs, tracings, collection and updating of cartographic information, navigation maps, tourist routes, etc [1, 2, 6, 11, 12, 15, 16].

Everyone who uses a projection coordinate system should know in which geodetic reference system it is implemented, the type of the projection system and its secondary parameters. Without knowing these parameters, a number of transformation problems related to the use of coordinates in it cannot be solved. For example, for BGS 2005, the parametric representation of the coordinate system implemented in GRS80 UTM will look like this:

- Reference system GRS80;
- Projection system UTM;
  - 6^0 zones numbered by meridian $L = 180^0$ east according to the formula: $n = (180 + L_0 + 3)/6$;
  - Scale along the prime meridian $m_0 = 0,9996$;
  - Added value to residents +500 000 m.

In the UTM coordinate system, the coordinate origin, in this case – the beginning of the stripes, is not set, unlike the conical Lambert projection, because it is defined in the projection itself as the intersection of the prime meridian with the Equator.

The scale values *m* for Gaussian (Table 6) and UTM (Table 7) projections are calculated using the following formula

$$\begin{aligned} m &= 1 + 0,0001523.\lambda^2.\cos^2\varphi.\left(1+\eta^2\right), \\ m_{UTM} &= m_0.m, \quad \text{where } m_0 = 0,9996 \end{aligned} \tag{16}$$

In cartographic practice, when calculating scales, the magnitude $\eta^2 = e^{`2}.\cos^2\varphi$ usually, it is not taken into account, since its value has negligible influence, the formula will take on the following form:

$$m = 1 + 0,0001523.\lambda^2.\cos^2\varphi. \tag{17}$$

For the territory of Bulgaria we will use the latitude of the middle parallel – $\varphi_0 = 42^0 40`00``$.

Table 6. Scale values

Меридиан (λ)	$0^0 30`$	1^0	$1^0 30`$	2^0	$2^0 30`$	3^0
Мащаб (m)	1,000021	1,000082	1,000185	1,000329	1,000515	1,000741

Table 7. Scale values

Меридиан (λ)	$0^0 30`$	1^0	$1^0 30`$	2^0	$2^0 30`$	3^0
Мащаб (m)	0,999621	0,999682	0,999785	0,999929	1,000114	1,000341

If we have a distance of 1000 m between two points, and the points are at the end of the six-degree strip, then a distance equal to 1000,74 m in the Gaussian projection and a distance equal to 1000,34 m in the UTM projection will be plotted in the map plane. Therefore, this will mean that we will have an extension of 74 cm and 34 cm respectively.

With the Lambert projection for the final parallels, we will get the following values (we use the data from Table 2 and 3) for two standard parallels – 1000,13 m, and for one standard parallel – 1000,34 m, which will mean that we will have an extension of 13 cm and 34 cm respectively. The values between the two standard parallels are 999,88 m, which will mean that we will have a shortening of 12 cm respectively.

4. Conclusion

For a mapped area, the scale m should be close to one. The minimum scale difference from one is an important sign of optimality, but it is also important that the transformation from the image (map) coordinates (x, y) to the ellipsoidal geographic coordinates (φ, λ) is easy and convenient.

From the analysis of the Gaussian projection and its modification UTM, and the results obtained for the Lambert projection, we can draw the following conclusion: For the territory of Bulgaria, a more suitable geodetic projection would be a conformal conic projection with two standard parallels.

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