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ALGORITHM CONSTRUCTION AND SOLVING OF OPTIMAL MANAGEMENT PROBLRM OF FINANCIAL SYSTEM

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Abstract. *The algorithm for solving of dynamic management problem is examined, that is conducted in a few stages: from discrete initial data we get optimal functional dependence; we find differential equalization of process; we set a management function and criterion of optimality; we conduct the numeral solution of problem. For a solving the method of the Pontryagin maximum principle is used. Equalization of the state we write regarding Gross Domestic Product. An optimal management of United Kingdom system state from initial to the set end position is offered.*

Keywords: *investment, management, optimality, profit, GDP.*
JEL classification codes: *C01; C53; E17; F21*

Introduction

A management problem of financial flows is finding differential function of state, set of management function and optimality criterion as well.

One method of solving the management problem is the method of the Pontryagin maximum principle [1,2,3,4].

The process of finding the optimal functional relationship between the variables can be found in [5,6,7].

At modelling of monotone processes when the amount of unknowns is insignificant, as research following nine functions can be used. These associations possess such property, that if separate values of variables X and Y satisfy to one of the equations average values also to it satisfy. For each of functions there are characteristic averages which can be arithmetical, geometrical and harmonic averages in this case. Correspondence of investigated functions and their average magnitudes is reduced in table 1. In this table $a, b = const$.

Table 1. Aspect of the average magnitudes characterising functions of a regression

№	Function aspect	Characteristic averages	
		$\bar{X}$	$\bar{Y}$
1	$Y = a + bX$	$\bar{X} = \sum_{i=1}^{n} X_i / n$	$\bar{Y} = \sum_{i=1}^{n} Y_i / n$
2	$Y = a + b\ln X$	$\bar{X} = \sqrt[n]{X_1 X_2 ... X_n}$	$\bar{Y} = \sum_{i=1}^{n} Y_i / n$
3	$Y = a + b / X$	$\bar{X} = n / \sum_{i=1}^{n} (1 / X_i)$	$\bar{Y} = \sum_{i=1}^{n} Y_i / n$
4	$Y = ab^X$	$\bar{X} = \sum_{i=1}^{n} X_i / n$	$\bar{Y} = \sqrt[n]{Y_1 Y_2 ... Y_n}$
5	$Y = aX^b$	$\bar{X} = \sqrt[n]{X_1 X_2 ... X_n}$	$\bar{Y} = \sqrt[n]{Y_1 Y_2 ... Y_n}$
6	$Y = \exp(a + b / X)$	$\bar{X} = n / \sum_{i=1}^{n} (1 / X_i)$	$\bar{Y} = \sqrt[n]{Y_1 Y_2 ... Y_n}$
7	$Y = 1/(a + bX)$	$\bar{X} = \sum_{i=1}^{n} X_i / n$	$\bar{Y} = n / \sum_{i=1}^{n} (1 / Y_i)$
8	$Y = 1/(a + b\ln X)$	$\bar{X} = \sqrt[n]{X_1 X_2 ... X_n}$	$\bar{Y} = n / \sum_{i=1}^{n} (1 / Y_i)$
9	$Y = X /(a + bX)$	$\bar{X} = n / \sum_{i=1}^{n} (1 / X_i)$	$\bar{Y} = n / \sum_{i=1}^{n} (1 / Y_i)$

Determination of one optimum function happens in some stages. At the first stage the necessary average magnitudes for variables X and Y are calculated. At the second stage, depending on $X_i < \bar{X} < X_{i+1}$, by means of linear interpolation values are calculated

$$\hat{Y}_i = Y_i + \frac{Y_{i+1} - Y_i}{X_{i+1} - X_i} (\bar{X} - X_i) \qquad (1)$$

At the third stage it is defined one of nine functions which in the best way describes input datas. As criterion of selection it is possible to use a condition

$$\left| \frac{\hat{Y} - \bar{Y}}{\hat{Y}} \right| \to \min \qquad (2)$$

The unknown constants, which are in the regression equation, are calculated by means of a method of least squares. This method is a basis of the regression analysis and consists of performance of a following condition for function of errors

$$S = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 \to \min \qquad (3)$$

Determination of an extremum (3) for linear regression functions, is reduced to a solution of linear system of the algebraic equations concerning parametres a and b. It is proved, that this system has a unique solution and function of errors S reaches the minimum. For application of a method of least squares to all regression functions (table 1), it is necessary to transform them beforehand. This transformation consists in their information to a linear aspect. Unknown constants which are calculated from a condition (3) definitively define an aspect of the optimum equation of a regression.

It is possible to continue modelling of initial process and receive the concrete differential equation which maps an investigated appearance. The constants entering into this differential equation are directly connected with the constants entering in the regression equation. In table 2 we will reduce values of constants a and b, which are calculated from condition (3), and also a corresponding boundary value problem for every regression equations. In this table following labels numerical magnitudes are used: $M(X)$ - expectation; $D(X)$ - a variance; $K(XY) = M(XY) - M(X)M(Y)$ - the correlative moment. Thus, knowing the optimum regression formula it is possible to construct corresponding mathematical model in the form of the differential equation.

Table 2. - Correspondence of the regression formula and a boundary value problem of Cauchy

№	Function aspect	Parametres a and b	The differential equation	Boundary condition
1	$Y = a + bX$	$a = M(Y) - bM(X)$ $b = K[XY]/D(X)$	$Y' = b$	$Y(1) = a + b$
2	$Y = a + b\ln X$	$a = M(Y) - bM(\ln X)$ $b = K[\ln(X)Y]/D(\ln X)$	$Y'X = b$	$Y(1) = a$
3	$Y = a + b/X$	$a = M(Y) - bM(X^{-1})$ $b = K[X^{-1}Y]/D(X^{-1})$	$Y'X^2 = -b$	$Y(1) = a + b$
4	$Y = ab^X$	$\ln a = M(\ln Y) - \ln bM(X)$ $\ln b = K[X \ln Y]/D(X)$	$Y'/Y = \ln b$	$Y(1) = ab$
5	$Y = aX^b$	$\ln a = M(\ln Y) - bM(\ln X)$ $b = K[\ln X \ln Y]/D(\ln X)$	$Y'X/Y = b$	$Y(1) = a$

6	$Y = \exp(a + b / X)$	$a = M(\ln Y) - bM(X^{-1})$ $b = K[X^{-1} \ln Y] / D(X^{-1})$	$Y'X^2 / Y = -b$	$Y(1) = \exp(a + b)$
7	$Y = 1/(a + bX)$	$a = M(Y^{-1}) - bM(X)$ $b = K[XY^{-1}] / D(X)$	$Y' / Y^2 = -b$	$Y(1) = 1/(a + b)$
8	$Y = 1/(a + b \ln X)$	$a = M(Y^{-1}) - bM(\ln X)$ $b = K[\ln(X) Y^{-1}] / D(\ln X)$	$Y'X / Y^2 = -b$	$Y(1) = 1/a$
9	$Y = X/(a + bX)$	$b = M(Y^{-1}) - aM(X^{-1})$ $a = K[X^{-1} Y^{-1}] / D(X^{-1})$	$Y'X^2 / Y^2 = a$	$Y(1) = 1/(a + b)$

Quality of stochastic connection between variables Y and X (quality of the regression equations) can be estimated by means of factor of determination which is calculated under the formula

$$R^2 = 1 - \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 / \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \qquad (4)$$

The formula (4) shows, how percent from the general variance of variable Y explains investigated the regression equation

Problem statement

As a function of the equation of state will use the Gross Domestic Product (GDP). Depending on the processed raw data we obtain the corresponding equation of state and initial conditions (Table 2). Assume that $X = t$.

As an example, consider the problem of optimal management with the initial data for United Kingdom [8] on the interval $[t_0; T]$:

Table 3: Initial data for calculation

t	Year	GDP (Current Prices, US$ Billion), *Y(t)*	Investment (% of GDP), $\alpha_1(t)$	General government total expenditure (% of GDP), $\alpha_2(t)$	General government revenue (% of GDP), $\alpha_3(t)$
1	2011	2471,88	15,584	45,861	37,301
2	2012	2602,49	16,411	44,48	37,542
3	2013	2743,35	17,358	42,793	37,776
4	2014	2890,99	18,18	41,26	37,843
5	2015	3050,52	18,913	40,072	37,779
6	2016	3220,42	19,601	38,704	37,384

Processing the data in Table 3, we find the optimal function in the form $Y = ab^t$. Applying the method of the least squares, we obtain the unknown parameters: $a = 2342{,}0198$; $b = 1{,}053276$. This optimal function corresponds to the following boundary value problem:

$$\frac{dY(t)}{dt} = \ln bY(t)\,;\, Y(t_0) = ab \qquad (5)$$

Equation (5) is the equation of studied system state, with the initial condition.

Find the optimal functional dependence for $\alpha_2(t)$- general government total expenditure (% of GDP) on the interval $t \in [t_0;T]$:

$$\alpha_2(t) = 47{,}2492 - 1{,}444057t \qquad (6)$$

Thus, we obtain a mathematical formulation of the overall cost management problem for the state

We will consider raising and solution of a few optimization problems.

Problem A

The equation of state of the system:

$$\frac{dY(t)}{dt} = \ln bY(t) + U(t) \qquad (7)$$

Initial conditions:

$$Y(t_0) = ab \qquad (8)$$

Management function has following form:

$$U(t) = \alpha_2(t)ab^t \qquad (9)$$

where $\alpha_2(t)$ has the form (6).

Optimality condition is maximization the discounted difference between the values of investment (% GDP) -$\alpha_1(t)$, general government revenue (% GDP) - $\alpha_3(t)$ and general government total expenditure (% GDP) - $\alpha_2(t)$ for a certain period and has the form:

$$\int_{t_0}^{T} \exp(-\delta t)(\alpha_1(t) + \alpha_3(t) - \alpha_2(t))Y(t)dt \to \max \qquad (10)$$

where δ - discount coefficient; $\alpha_1(t), \alpha_3(t) > 0$.

Thus, it is necessary to maximize the integral equation (10) when the conditions (7) - (9) are fulfilled.

To solve the problem we apply the Pontryagin maximum method [3].

We write the Hamiltonian function:

$$H(t) = \Psi(t)\{\alpha_2(t)ab^t + \ln bY(t)\} + \exp(-\delta t)\{\alpha_1(t) + \alpha_3(t) - \alpha_2(t)\}Y(t) \tag{11}$$

where $\Psi(t)$ - auxiliary function that satisfy equation

$$\frac{d\Psi(t)}{dt} = -\Psi(t)\ln b - \exp(-\delta t)\{\alpha_1(t) + \alpha_3(t) - \alpha_2(t)\} \tag{12}$$

For the auxiliary function transversality is carried out

$$\Psi(T) = 0$$

By analyzing the Hamiltonian (12) we obtain an optimal investment strategy:

$$\alpha_2(t) = \begin{cases} \max \alpha_2(t) = \alpha_2(t_0)\ , t_0 \le t \le t_1^* \\ \min \alpha_2(t) = \alpha_2(T) \ \ , t_1^* < t \le T \end{cases} \tag{13}$$

where t_1^* - time of switching company's investments, which is found from condition

$$\Psi(t_1^*)ab^{t_1^*} - \exp(-\delta t_1^*)Y(t_1^*) = 0 \tag{14}$$

As a function $Y(t)$ we use the solution of boundary problem (7), (8) with the function of management in a form (9). Assume that $\alpha_2(t) = const$.

$$Y(t) = (1 - \alpha_2 - \alpha_2 t)ab^t \tag{15}$$

Auxiliary variable $\Psi(t)$ in the interval $[t_0; t_1^*]$ in the management $\alpha_2(t) = \alpha_2(t_0)$ is determined by solving the boundary problem:

$$\begin{cases} \dfrac{d\Psi(t)}{dt} = -\Psi(t)\ln b - \exp(-\delta t)(\alpha_1(t) + \alpha_3(t) - \alpha_2(t_0)) \\ \Psi(t_1^*) = \exp(-\delta t_1^*)\{1 - \alpha_2(t_0) - \alpha_2(t_0)t_1^*\} \end{cases} \tag{16}$$

Solution (16) assuming

$$\begin{aligned} &\alpha_1(t) = \max \alpha_1(t) = \alpha_1 = const \\ &\alpha_3(t) = \max \alpha_3(t) = \alpha_3 = const \end{aligned} \tag{17}$$

Has the form:

$$\Psi(t) = \exp(-\delta t_1^*)\exp(\ln b(t - t_1^*))\{1 - \alpha_2(t_0) - \alpha_2(t_0)t_1^* - C\} + C\exp(-\delta t) \qquad (18)$$

where $C = \dfrac{\alpha_1 + \alpha_3 - \alpha_2(t_0)}{\delta - \ln b}$

On interval $[t_1^*; T]$ with the management $\alpha_2(t) = \alpha_2(T)$ function $\Psi(t)$ is determined by solving the problem:

$$\begin{cases} \dfrac{d\Psi(t)}{dt} = -\Psi(t)\ln b - \exp(-\delta t)(\alpha_1(t) + \alpha_3(t) - \alpha_2(T)) \\ \Psi(T) = 0 \end{cases} \qquad (19)$$

The solution (19) with constraints (17) has the form:

$$\Psi(t) = D\{\exp(-\delta t) - \exp[-T(\ln b - \delta) - t\ln b]\} \qquad (20)$$

where $D = \dfrac{\alpha_1 + \alpha_3 - \alpha_2(T)}{\delta - \ln b}$

Time of switching company's investments is obtained from condition (14)

$$\exp(-\delta t_1^*)\{D - 1 + \alpha_2(t_0) - \alpha_2(t_0)t_1^*\} = \exp\{T(\ln b - \delta) - t_1^*\ln b\}D \qquad (21)$$

Numerical examples

As a numeral experiment following basic data is used:

$\delta = 0{,}01$; $\alpha_1 = 0{,}19601$; $\alpha_3 = 0{,}37843$; $t_0 = 1$; $T = 6$; $Y(1) = 2466{,}616$

While solving equalization (21), we find a switchpoint $t_1^* = 1{,}065$. Optimal distribution of management function comes true taking into account a switchpoint.

Problem B

Equalization of the state, initial conditions and management function look like (7) - (9) accordingly, and the condition of optimality we will take in a form:

$$\int_{t_0}^{T} \exp(-\delta t)\{(\alpha_1(t) + \alpha_3(t))Y(t) - \alpha_2(t)ab^t\}dt \to \max \qquad (22)$$

To solve the problem we apply the Pontryagin maximum method [3].

We write the Hamiltonian function:

$$H(t) = \Psi(t)\{\alpha_2(t)ab^t + \ln bY(t)\} + \exp(-\delta t)\{(\alpha_1(t) + \alpha_3(t))Y(t) - \alpha_2(t)ab^t\} \quad (23)$$

where $\Psi(t)$ - auxiliary function that satisfy equation

$$\frac{d\Psi(t)}{dt} = -\Psi(t)\ln b - \exp(-\delta t)\{\alpha_1(t) + \alpha_3(t)\} \quad (24)$$

For the auxiliary function transversality is carried out

$\Psi(T) = 0$

By analyzing the Hamiltonian (24) we obtain an optimal investment strategy (13):

Time of switching company's investments, which is found from condition

$$\Psi(t_1^*) - \exp(-\delta t_1^*) = 0 \quad (25)$$

Auxiliary variable $\Psi(t)$ in the interval $[t_0; t_1^*]$ in the management $\alpha_2(t) = \alpha_2(t_0)$ is determined by solving the boundary problem:

$$\begin{cases} \dfrac{d\Psi(t)}{dt} = -\Psi(t)\ln b - \exp(-\delta t)(\alpha_1(t) + \alpha_3(t)) \\ \Psi(t_1^*) = \exp(-\delta t_1^*) \end{cases} \quad (26)$$

Solution (26) assuming (17)

Has the form:

$$\Psi(t) = \exp(-\delta t_1^*)\exp(\ln b(t - t_1^*))(1 - E) + E\exp(-\delta t) \quad (27)$$

where $E = \dfrac{\alpha_1 + \alpha_3}{\delta - \ln b}$

On interval $[t_1^*; T]$ with the management $\alpha_2(t) = \alpha_2(T)$ function $\Psi(t)$ is determined by solving the problem:

$$\begin{cases} \dfrac{d\Psi(t)}{dt} = -\Psi(t)\ln b - \exp(-\delta t)(\alpha_1(t) + \alpha_3(t)) \\ \Psi(T) = 0 \end{cases} \quad (28)$$

The solution (28) with constraints (17) has the form:

$$\Psi(t) = E\{\exp(-\delta t) - \exp[-T(\ln b - \delta) - t \ln b]\} \tag{29}$$

Time of switching company's investments is obtained from condition (15)

$$t_1^* = T - \frac{1}{\ln b - \delta} \ln(\frac{E-1}{E}) \tag{30}$$

Numerical examples

As a numeral experiment we will set following initial problem data of A.

While solving equalization (30), we find a switchpoint $t_1^* = 4{,}32$. Thus, for the founding of optimal solution it is necessary to take into account the found switchpoint. We will notice that a switchpoint in the problem A is before, than in the problrm B at the same initial conditions.

Conclusion

The algorithm of initial problem construction is used for a further analysis. Optimal functions and corresponding to them boundary problem are got. A solution of a few optimal management problem is shown through the method of the Pontryagin maximum principle.

References

[1]. Optimal Control in Economics: Theory and Applications: Training. posobie.-/B.A.Logosha, T.G.Apalkova.-M.: Finance and Statistics, 2008.-P.224

[2]. Arutyunov A.V. Pontryagin Maximum Principle. Proofs and applications / A.V.Arutyunov, G.G.Magaril-Il'yaev V.M.Tihomirov.-M. Univ "Factorial Press", 2006. - P.144

[3]. Pontryagin L.S, Boltyanskii V.G, Gamkrelidze R.V., Mishchenko E.F. The mathematical theory of optimal processes. Nauka,1983.-P. 392

[4]. Shell K. Application of Pontryagin's maximum principle to economics // Mathematical system theory and economics. I.Berlin: Springer,1969.P.241-292 (Leet.Notes Oper. Res. And Math.Econ.;V.11).

[5]. Oliynyk V.Construction of economic models The 1 st International Conference on Applied Business and Economics (ICABE 2009), Faculty of Business, Sohar Universyity, Sohar, Sultanate of Oman, 30-31 March,2009. - p.p.258-263

[6]. Lvovskiy E.N. Statistical methods for construction of empirical formulas. Vysshaya Shkola, 1988.-P.239.

[7]. Nazarenko A.M. Econometrics. - Sumy: SSU, 2003.-P.271

[8]. http://www.economywatch.com/economic-statistics/