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QUASI-MONTE CARLO METHODS BASED ON SOBOL AND HALTON SEQUENCES FOR COMPUTATION OF MULTIDIMENSIONAL INTEGRALS APPLIED IN SECURITY SYSTEMS

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ABSTRACT: *In this paper we implement and analyze the performance of Sobol quasi-random sequence and compare the results with the Halton quasi-random sequence has been done. We consider a case study with a non-smooth integrand function with applications in financial mathematics. We show that the Sobol sequence has some advantageous over the Halton sequence. It is established that Sobol sequence performs better than other sequences for higher dimensions.*

KEYWORDS: *Financial mathematics, Quasi-Monte Carlo algorithms, multidimensional integrals, Halton sequence, Sobol sequence, Computational complexity*

Introduction

Monte Carlo methods give statistical estimates for the functional of the solution by performing random sampling of a certain random variable whose mathematical expectation is the desired functional. Monte Carlo methods are methods of approximation of the solution to problems of computational mathematics, by using random processes for each such problem, with the parameters of the process equal to the solution of the problem. High dimensional integrals are usually solved with Monte Carlo algorithms and quasi

Monte Carlo algorithms. Monte Carlo method is the only viable method for high-dimensional problems since its convergence is independent of the dimension. The method can guarantee that the error of Monte Carlo approximation is smaller than a given value with a certain probability [5]. The most important advantage of the Monte Carlo methods is that they are suitable for solving multi-dimensional problems, since the computational complexity increases linearly and not exponentially with the dimensionality [5,14]. The MC method is a widely used tool in many fields of science.

The Monte Carlo encompasses any technique of statistical sampling employed to approximate solutions to quantitative problems. Essentially, the Monte Carlo method solves a problem by directly simulating the underlying (physical) process and then calculating the (average) result of the process. This very general approach is valid in areas such as physics, chemistry, etc [8].

In finance, the Monte Carlo method is used to simulate the various sources of uncertainty that affect the value of the instrument, portfolio and investment in question, and to then calculate a representative value given these possible values of the underlying inputs [6,7].

In the last few years new approaches in finance have been developed that outperform standard Monte Carlo in terms of numerical efficiency. It has been found that there can be efficiency gains in using deterministic sequences rather than the random sequences which are a feature of standard Monte Carlo. These deterministic sequences are carefully selected so that they are well dispersed throughout the region of integration. Sequences with this property are known as low discrepancy sequences. These sequences are often more efficient than standard Monte Carlo in evaluating high dimensional integrals if the integrand is sufficiently regular and for many finance applications this is the case. However it is interesting to consider multidimensional integrals of non-smooth function. Such integrals can be used to describe problems in option pricing in finance, where multidimensional integrals are used for statistical measures for option pricing [19].

The Monte Carlo method is known to be only accurate with a tremendous amount of scenarios since its rate of convergence is $O(N^{-1/2})$. Quasi Monte Carlo methods use deterministic sequences that have better uniform properties measured by discrepancy. They are usually superior to the Monte Carlo method as they have a convergence rate of $((\log N)^d/N)$, where N is the number of samples and d is the dimensionality of the problem under consideration.

Quasi Monte Carlo algorithms for numerical integration

Consider the problem of approximate integration of the multiple integral of dimension d.

For small values of d, numerical integration methods such as Simpson's rule or the trapezoidal rule can be used to approximate the integral. These methods, however, suffer from the so-called curse of dimensionality and become impractical as d increases.

The crude Monte Carlo method has rate of convergence $O(N^{-1/2})$ which is independent of the dimension of the integral, and that is why Monte Carlo integration is the only practical method for many high-dimensional problems. Much of the efforts to improve Monte Carlo are in construction of variance reduction methods which speed up the computation or to use quasi-random sequences [5,12,15]. A quasi-random or low discrepancy sequence, such as the Faure, Halton, Hammersley, Niederreiter or Sobol sequences [19,20,21], is "less random" than a pseudorandom number sequence, but more useful for such tasks as approximation of integrals in higher dimensions, and in global optimization. This is because low discrepancy sequences tend to sample space "more uniformly" than random numbers. It is a question of interest to know which sequence outperforms the other. In this study we implement the Halton sequence and make a comparison with the Sobol sequence.

Quasi Monte Carlo algorithm based on Halton sequence

The Halton sequences [9,10] are a digital (0, s)-sequence over F_b with b denoting a prime (original case) or a prime power (general case) greater or equal to d. The Halton sequence is constructed according to a deterministic method that uses coprime numbers as its bases. As a simple example, let's take one dimension of the Halton sequence to be based on 2 and the other on 3. The algorithm for the Halton sequence follows the method of John Halton in [9,11] for computing quasi-random numbers. We use of persistent variables to improve the MATLAB implementation. The parameters of the Halton algorithm are described below. The input, integer I1, I2, the indices of the first and last elements of the sequence, 0 <= I1, I2. We also have an input, integer M, the spatial dimension,1 <= M <= 100. The output is a real R(M,abs(I1-I2)+1), the elements of the sequence with indices I1 through I2. It is important to mention that the Halton sequence used here is not scrambled, a comparison with a scrambled sequence will be an object of a fututre study. A comparison with the Sobol sequnce [18,19,20] has been made for the first time for a special kind of multidimensional integral with non smooth integrand function used in finance [20,21].

Quasi Monte Carlo algorithm based on Sobol sequence

For the Sobol sequence we use an implementation that is an adapation of the INSOBL and GOSOBL routines in ACM TOMS Algorithm 647 [9] and ACM TOMS Algorithm [1,2,3,4]. The original code can only compute the "next" element of the sequence [11]. The revised code allows the user to specify the index of the desired element. The algorithm has a maximum spatial dimension of

40 since MATLAB doesn't support 64 bit integers. A remark by Joe and Kuo [10] shows how to extend the algorithm from the original maximum spatial dimension of 40 up to a maximum spatial dimension of 1111. The FORTRAN90 and C++ versions of the code has been updated in this way [12,15,19], but updating the MATLAB code has not been simple, since MATLAB doesn't support 64 bit integers. We use algorithm that generates a new quasi-random Sobol vector with each call. The routine adapts the ideas of Antonov and Saleev [1,13,22] . The parameters of the algorithm are an integer DIM_{NUM}, the number of spatial dimensions. The algorithm starts with integer *SEED*, the "seed" for the sequence. This is essentially the index in the sequence of the quasi-random value to be generated. On output, *SEED* has been set to the appropriate next value, usually simply *SEED* + 1. If *SEED* is less than 0 on input, it is treated as though it were 0. An input value of 0 requests the first (0-th) element of the sequence. Output is the real *QUASI*(*DIMNUM*), the next quasi-random vector [1,5.15].

Numerical example and results

Table 1: The relative error for 4 dimensional integral of a non-smooth function with Faure and Sobol QMC

N	Halton	Time,s	Sobol	Time,s
10^2	3.15e-2	0.03	5.67e-3	0.02
10^3	6.32e-3	0.43	8.91e-4	0.21
10^4	9.10e-4	3.93	5.51e-4	1.78
10^5	1.97e-4	42.3	5.79e-5	18.6
10^6	5.09e-5	200	7.70e-6	119

Table 3: The computational time for 4 dimensional integral of a non-smooth function with the three methods

Time, s	Halton	Sobol
0.1	1.32e-2	1.86e-3
1	3.46e-3	7.15e-4
5	3.26e-4	2.03e-4
10	1.22e-4	9.75e-5
60	7.91e-5	8.91e-6

We will test Halton and Sobol sequence for evaluating the following multidimensional integral of non-smooth integrand function, which is:

$$f_1(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} |(x_i - 0.8)^{-1/3}|,$$

where the integration is over the unit 4-dimensional hypercube. The referent value of the integral is 7.22261 [5,16]. exist. Such applications appear also in some important problems in financial

mathematics.

As can been seen from the table for 4 dimensional integral, the low discrepancy sequence of Sobol produces more rapid convergence, and

lower errors, than the Halton sequence. It is important to mention that the the pseudo randomly obtained averages converge at the rate $O(N{-}1/2)$, while the quasi randomly obtained averages converge at a rate closer to $O(N^{-1})$. As expected the Sobol sequence gives better results than the Halton sequence. It is interesting to see that for a given number of samples the Halton sequence gives better relative error than the Sobol – see Table 1, and the Sobol sequence is two times faster. It is interesting that for a preliminary given time up to 5s the Sobol and Halton gives closer relative errors – see Table 2, but for 60s the Sobol sequence starts producing better results. For a preliminary given time of only 1s Sobol sequence gives relative error of 7.15e-4 and Halton sequence gives relative error of 3.46e-3 which is already a sufficient accuracy. Definitely for larger number of points and for a preliminary given time, the advantage of Sobol quasi-random sequence in place of both Halton sequence should become even more pronounced – see Table 1 and Table 2. For this example of non-smooth integrand function arises in financial mathematics even the first derivative does not exist.

Conclusion

In this paper we analyze the performance of two quasi Monte Carlo methods for multidimensional integrals. The Sobol quasi-random sequence is compared with the Halton sequence and the results are very precise for the multidimensional integrals under consideration, which shows the strength of the presented algorithm for relatively low dimensions. We consider an example of non-smooth integrand functionwih applications in finance. For this particular example the stochastic algorithms under consideration produce reliable results. While Sobol sequence is better than the Halton sequence. In the future a scrambled version of the two algorithms will be presented. The experiments show that for a preliminary given time the Sobol algorithm outperform the Halton sequence. The multidimensional integral under consideration is with non-smooth integrand which is more difficult case than the smooth integrand function. Stochastic methods under consideration are an efficient way to solve problems like the mentioned above that are described with multidimensional integrals of both smooth and non-smooth functions. It will be interesting to compare Faure sequence with Halton and Niederreiter sequences, but this will be an object of a future study which can find appliance in security systems.

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